JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 56, 317–329 (1976)

Perturbation Algebra of an Elliptic Operator

S. I. ROSENCRANS

Tulane University, New Orleans, Louisiana 70118

Submitted by Gian-Carlo Rota

1. INTRODUCTION

In this paper we explain how to use the group properties of parabolic equations $u_t = Lu$ to solve certain perturbed equations $w_t = (L + P)\,w$ by a process of quadrature from the former. The perturbations P form a Lie algebra $\mathscr{P}$ isomorphic to a subalgebra of the algebra $\mathscr{G}$ corresponding to $u_t = Lu$; $\dim \mathscr{P} = \dim \mathscr{G} - 1$. In effect second-order elliptic operators are divided into equivalence classes; within each class $\{L(\alpha)\}$ solutions to the Cauchy problem $u(\alpha)_t = L(\alpha)\,u(\alpha)$ with fixed data ϕ (independent of α) can be obtained by quadrature from any one solution $u(\alpha_0)$.

This should be contrasted with the usual application of Lie groups to differential equations; namely, to the discovery of particular solutions. Some references for the latter topic are [1, 2, 8, 18–20]. The group of the heat equation is derived in [1, 16]; in [9], though incompletely; in [22], after some changes of variables. References [12–17] apply the theory to explain the phenomenon of separation of variables.

We are motivated by the following trick. The heat equation,

$$\partial u/\partial t = D^2 u \qquad (D = \partial/\partial x)$$

is invariant under the transformations $x \mapsto \exp(s)\,x$, $t \mapsto \exp(2s)\,t$, and hence if u is any solution (say $\exp(tD^2)\,\phi$) so is $u(\exp(2s)\,t, \exp(s)\,x)$; its initial value is $\phi(\exp(s)\,x) \equiv \exp(sxD)\,\phi$. This can be summarized by the equation[1]

$$e^{sxD} e^{\exp(2s)tD^2} = e^{tD^2} e^{sxD}. \tag{1}$$

[1] Another explanation for (1), at a formal Lie-algebraic level, is that the set of multiples of D^2 is an ideal in the two-dimensional Lie algebra generated by D^2 and xD (see Table I). Hence elements of the form $\exp(rD^2)$ form a local normal subgroup in the corresponding Lie group, from which (1) follows, except for the determination of the coefficient $t\exp(2s)$; this can be accomplished by using the adjoint representation. Here $\exp(rD^2)$ stands *not* for a semigroup of operators but for a Lie group element.

(*Warning.* The exponentials involving D^2 are merely a symbolic expression for the semigroups of operators so generated; $\exp(sxD)$ can be regarded in this light, but also can be thought of as the usual exponential map of a Lie derivative.)

We use (1) to uncover a one-parameter subgroup of the two-parameter (semi)-group

$$(t, s) \to e^{tD^2} e^{sxD}.$$

Using (1), we find that

$$(t, s) \cdot (t_1, s_1) = (t_1 e^{-2s} + t, s + s_1).$$

Some simple manipulations yield the one-parameter subgroup $t = a(\exp(2cr) - 1)$, $s = -cr$, for constants a and c and subgroup parameter r. Therefore for $r \geqslant 0$ ($a > 0$, $c > 0$)

$$\begin{aligned} H(r) &= e^{a(\exp(2cr)-1)D^2} e^{-crxD} \\ &= e^{-crxD} e^{a(1-\exp(-2cr))D^2} \end{aligned} \tag{2}$$

is a one-parameter semigroup of operators (on some appropriate function space; for example, bounded continuous functions). The formal generator is easily computed, with the result

$$\lim_{r \downarrow 0} \frac{H(r)\phi - \phi}{r} = 2acD^2\phi - cxD\phi \qquad (\phi \text{ smooth})$$

so (2) implies that the bounded solution to the Cauchy problem for the heat equation with linear drift

$$\begin{aligned} \partial w/\partial t &= 2ac(\partial^2 w/\partial x^2) - cx(\partial w/\partial x) \\ w(0, x) &= \phi(x) \qquad \text{(bounded)} \end{aligned} \tag{3}$$

is $w = u(a(1 - \exp(-2ct)), \exp(-ct)\, x)$, where u is the bounded solution to the problem

$$\begin{aligned} \partial u/\partial t &= \partial^2 u/\partial x^2 \\ u(0, x) &= \phi(x). \end{aligned}$$

(This fact is well known.)

In Section 2 we derive the necessary facts about the group of the diffusion equation $u_t = Lu$. The general method is set out in Sections 3 and 4, and applied to two examples in Section 5. One of the examples is the trick that was just outlined. The paper ends with a short table of perturbation algebras in Section 6.

We wish to point out that while the trick can be explained systematically by our theory, the theory is limited to linear equations, whereas the trick is not. For example, one can mimic the above calculations for Burgers' equation

$$u_t + uu_x = u_{xx}\,,$$

invariant under the group $s \to \exp(s)\, u(\exp(2s)\, t, \exp(s)\, x)$. One finds that that $w = \exp(-ct)\, u(a(1 - \exp(-2ct)), \exp(-ct)\, x)$ satisfies

$$w_t = 2acw_{xx} - 2acww_x - cxw_x - cw$$

with the same initial data as u.

2. Group of the Diffusion Equation

The treatment given here differs computationally from the development in [1, 2, 20], which is based upon consideration of the line and curvature bundles of a manifold; it is similar to the method of [11]. Let L be a nondegenerate elliptic operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)\, id.$$

The coefficients are assumed to be smooth and such that L generates an analytic semigroup of operators, denoted $\exp(tL)$, for at least small $t \geqslant 0$, on some locally convex space $\mathscr{X}$ of initial functions $\phi(x)$, and appropriate domain $\mathrm{Dom}(L)$. We do not bother to be any more specific because any particular choice of the space $\mathscr{X}$ would space upon the coefficients growth restrictions that are (by and large) unnatural as far as the Lie theory is concerned. The solution to

$$\begin{aligned} \partial u/\partial t &= Lu \\ u(0, x) &= \phi(x) \end{aligned} \tag{4}$$

is $u \equiv \exp(tL)\, \phi$. Let the set of solutions to the equation $u_t = Lu$ be denoted by $\mathscr{V}$. We seek to determine the local Lie transformation group $\mathscr{G}$ which transforms every element of $\mathscr{V}$ to another element of $\mathscr{V}$. Actually, it can be shown (see [21]) that with the exception of the trivial infinite-dimensional subgroup

$$u \mapsto u + \text{any fixed element of } \mathscr{V} \qquad (u \in \mathscr{V})$$

every transformation in $\mathscr{G}$ is of the form

$$u(t, x) \mapsto \nu(p(t, x))\, u(p(t, x)) \equiv v \qquad (u \in \mathscr{V}) \tag{5}$$

where $p \in G_1$, a transformation group acting on (t, x)-space. In effect G acts on $\mathscr{V}$ as a "multiplier" representation of G_1; see Miller [10, p. 17] for this concept. A typical generator of G will be

$$Z = \alpha(\partial/\partial t) + \beta \cdot \nabla + \gamma \, id$$

where α, β, γ are functions of (t, x); $\beta \cdot \nabla \equiv \Sigma\beta_i \, \partial/\partial x_i$. Correspondingly we have a local one-parameter group of solutions (suppressing (t, x)):

$$v(s) = \exp(sZ)\, u \qquad \text{(small real } s\text{)}.$$

This v is the unique solution to the first-order Cauchy problem

$$\begin{aligned} \partial v/\partial s &= \alpha(\partial v/\partial t) + \beta \cdot \nabla v + \gamma v \\ v(0) &= u. \end{aligned} \tag{6}$$

(The theory of characteristics applied to (6) leads to (5) with appropriate $p = p(s)$.)

It follows from the linearity of Eq. (4) that

$$\lim_{s \downarrow 0} \frac{e^{sZ}u - u}{s} = Zu \in \mathscr{V}$$

and, conversely, that if $Zu \in \mathscr{V}$ for some first-order partial differential operator Z then Z is one of the generators of G. This tells how to compute α, β, γ: they must be such that

$$\begin{aligned} &((\partial/\partial t) - L)\, u = 0 \\ \text{implies} \quad &((\partial/\partial t) - L)\,(\alpha u_t + \beta \cdot \nabla u + \gamma u) = 0. \end{aligned} \tag{7}$$

Equivalently,

$$\partial Z/\partial t = [L, Z] \qquad (\text{on } \mathscr{V}); \tag{8}$$

here $\partial Z/\partial t$ means $\alpha_t \, \partial/\partial t + \beta_t \cdot \nabla + \gamma_t id$; $[L, Z] = LZ - ZL$. It is obvious that the operators Z satisfying (8) from a Lie algebra under commutation.

PROPOSITION 1. *The coefficient α is independent of x, i.e., $\alpha = \alpha(t)$.*

PROPOSITION 2. *Every generator Z satisfies an ordinary differential equation*

$$a_l(\partial^l Z/\partial t^l) + a_{l-1}(\partial^{l-1} Z/\partial t^{l-1}) + \cdots + a_0 Z = 0,$$

for some constants $a_0, \ldots, a_l$, and integer $l \leqslant \dim G$.

Proofs. From (7) you find that

$$\alpha L^2 u - L(\alpha L u) + \beta \cdot \nabla(Lu) - L(\beta \cdot \nabla u)$$
$$= \text{terms involving spatial derivatives of } u \text{ of order} \leqslant 2,$$

for all $u \in \mathscr{V}$. Hence in particular, the third-order derivatives occurring on the left must add up to zero. A long calculation yields

$$\sum_{i,j,k,m} a_{ij} a_{km} \frac{\partial \alpha}{\partial x_j} \frac{\partial^3 u}{\partial x_i\, \partial x_k\, \partial x_m} = 0 \qquad \text{all } (t, x). \tag{9}$$

In particular, choosing $u(0, x) = \phi(x)$ with

$$\frac{\partial^3 \phi}{\partial x_i\, \partial x_k\, \partial x_m} = \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} \frac{\partial \alpha}{\partial x_m} \qquad \text{at a fixed } x_0$$

we see that (9) at $t = 0$ reduces to

$$A(\nabla \alpha \mid_{x_0})^2 = 0$$

where A is the quadratic form (at x_0) arising from L. Since A is positive definite, $\nabla \alpha$ must be zero at $t = 0$ for all x_0. The same is true of $\nabla(\partial^k \alpha / \partial t^k)$ at all $(0, x_0)$, as can be seen by differentiating (9) with respect to t several times. Proposition 2 then implies that $\nabla \alpha$ is zero for all $t \geqslant 0$. It remains to prove Proposition 2. This is obvious, though, because *every time derivative of* Z *also satisfies* (8) *and is hence a possible generator.*

3. Group Induced on the Data

If u is a solution to $u_t - Lu = 0$ then so is, at least for small s, $\exp(sZ)u$, but with some new data, say $R(s)\,\phi$, replacing the old data ϕ, i.e.,

$$e^{sZ} e^{tL} = e^{tL} R(s) \qquad \text{on} \qquad \mathscr{X} \tag{10}$$

(remember the warning given after Eq. (1))

$$R(s)\,\phi = \lim_{t \downarrow 0} v(s) \qquad \text{(see (6))}. \tag{11}$$

We are interested in examining this induced transformation in detail. In general $R(s)$ is a continuous linear transformation on $\mathscr{X}$ and $R(s)\,\phi \to \phi$ (in $\mathscr{X}$) as $s \downarrow 0$ (see (5) and (6)). Also

$$\begin{aligned} e^{tL} R(r)\, R(s)\, \phi &= e^{rZ} e^{tL} R(s)\, \phi \\ &= e^{rZ} e^{sZ} e^{tL} \phi \\ &= e^{(r+s)Z} e^{tL} \phi \\ &= e^{tL} R(r + s)\, \phi \end{aligned}$$

so $R(r)\,R(s) = R(r + s)$ for $r, s \geqslant 0$. Therefore $R(s)$: $s \geqslant 0$ has some generator M

$$M\phi = \lim_{s \downarrow 0} \frac{R(s)\,\phi - \phi}{s} \qquad \phi \in \mathrm{Dom}(M)$$

which can be computed from (6). Consider

$$\begin{aligned}
\frac{R(s)\,\phi - \phi}{s} &= \lim_{t \downarrow 0} e^{tL} \left(\frac{R(s)\,\phi - \phi}{s} \right) \\
&= \lim_{t \downarrow 0} \frac{e^{sZ}u - u}{s} \\
&= \lim_{t \downarrow 0} \frac{1}{s} \int_0^s \frac{\partial v}{\partial s_1}\, ds_1 \\
&= \lim_{t \downarrow 0} \frac{1}{s} \int_0^s Zv(s_1)\, ds_1 \\
&= \lim_{t \downarrow 0} \frac{1}{s} \int_0^s \left\{ \alpha(t) \frac{\partial v}{\partial t} + \beta(t, x) \cdot \nabla v + \gamma(t, x)\, v \right\} ds_1 \\
&= \lim_{t \downarrow 0} \frac{1}{s} \int_0^s \{ \alpha(t)\, Lv(s_1) + \beta \cdot \nabla v(s_1) + \gamma v(s_1) \}\, ds_1 \\
&= \frac{1}{s} \int_0^s \{ \alpha(0)\, LR(s_1)\,\phi + \beta(0, x) \cdot \nabla R(s_1)\,\phi + \gamma(0, x)\, R(s_1)\,\phi \}\, ds_1 .
\end{aligned}$$

The limit as $s \downarrow 0$ exists of and only if $\phi \in \mathrm{Dom}(L)$, so

$$M\phi = \alpha(0)\, L\phi + \beta(0, x) \cdot \nabla\phi + \gamma(0, x)\,\phi \qquad \phi \in \mathrm{Dom}(L). \tag{12}$$

Equations (10), (11), and (12) contain the main point, namely, that w, *the limit as* $t \downarrow 0$ *of the transformed solution* $v(s) = \exp(sZ)\, u$, *is itself the unique solution to the Cauchy problem*

$$\begin{aligned} \partial w / \partial s &= Mw \\ w(0) &= \phi \end{aligned} \tag{13}$$

with the same data ϕ *as before.*

Equation (10) should now be written as

$$e^{sZ} e^{tL} = e^{tL} e^{sM} \qquad \text{on} \qquad \mathscr{X} \tag{14}$$

or, equivalently,

$$Ze^{tL} = e^{tL} M \qquad \text{on} \qquad \mathrm{Dom}(L). \tag{15}$$

Recapitulation. To solve (13), which is a parabolic Cauchy problem like (4), but *with some extra* $\leqslant$ *first-order perturbations,* you solve (4) first, then (6) (this is merely a quadrature), and finally set $t = 0$. Examples will be given in Sections 5 and 6.

4. The Perturbation Algebra

Given an elliptic operator L, we inquire as to the character of the set of perturbations discussed in the last paragraph, i.e., the lower order perturbations allowing an integration by quadrature. We show that the perturbations form a Lie algebra $\mathscr{P}$ (under commutation) of one less dimension than that of the algebra of the original group G, i.e., one dimension less than that of the algebra of the Z's.

First of all, (12) shows that every Z induces a unique M. On the other hand letting $t \downarrow 0$ in (8) we find

$$\begin{aligned}[L, M] &= \alpha_t(0) L + \beta_t(0, x) \cdot \nabla + \gamma_t(0, x)\, id \\ &= \text{another allowable } M.\end{aligned} \tag{16}$$

(See the last sentence in the proof of Propositions 1 and 2.) Consequently, if M is given but Z is unknown, all l initial time derivatives of Z can be computed by repeated bracketing with L. By Proposition 2, Z is thus uniquely determined.

We must show that brackets are preserved in the passage from Z to M and back. Suppose $M_1 \leftrightarrow Z_1$ and $M_2 \leftrightarrow Z_2$. Then (use (15))

$$\begin{aligned} e^{tL}[M_1, M_2]\,\phi &= e^{tL} M_1 M_2 \phi - e^{tL} M_2 M_1 \phi \\ &= Z_1 e^{tL} M_2 \phi - Z_2 e^{tL} M_1 \phi \\ &= Z_1 Z_2 e^{tL} \phi - Z_1 Z_2 e^{tL} \phi \\ &= [Z_1, Z_2]\, u \end{aligned}$$

so $[M_1, M_2] \leftrightarrow [Z_1, Z_2]$.

The M's thus form a Lie algebra $\mathscr{M}$ isomorphic to that of the Z's. The *perturbations* (i.e., the residue from (12) when $\alpha(0) L$ is removed) are themselves elements of the algebra $\mathscr{M}$ because $L \in \mathscr{M}$: in fact L corresponds to the generator $Z = \partial/\partial t$, which generates a time translation and is always allowed because the coefficients of L do not depend on t. Finally, $\mathscr{P}$ is obviously closed under the bracket, as the commutator of two $\leqslant$ first-order differential operators, while in $\mathscr{M}$, cannot contain a second-order term $\alpha(0) L$. This completes the proof.

We now give a method for computing the perturbation algebra directly, without first computing the group G (or equivalently, the Z's). Of course, to carry out the quadratures, G must eventually be computed. But it is nice to have a quick way of finding the allowed perturbations.

PROPOSITION 3. *Suppose the coefficients of L to be analytic. A $\leqslant$ first-order differential operator P_0 is in the algebra $\mathscr{P}$ if and only if there is a sequence of $\leqslant$ first-order differential operators $P_1, P_2, \ldots$ and constants $\lambda_1, \lambda_2, \ldots$ such that*

$$[L, P_n] = \lambda_n L + P_{n+1} \qquad n \geqslant 0, \tag{17}$$

and $\sum \lambda_k t^k/k!$ and $\sum P_k t^k/k!$ converge at least for small t.

Proof. If $P_0 \in \mathscr{P}$ then (17) follows from (16) and (12). In this case

$$\lambda_n = \alpha^{(n+1)}(0)$$
$$P_{n+1} = \beta^{(n+1)}(0, x) \cdot \nabla + \gamma^{(n+1)}(0, x)\, id.$$

Since Z is analytic, the indicated series do converge.

In the converse direction, define

$$Z = \sum_{k=0}^{\infty} \lambda_k \frac{t^{k+1}}{(k+1)!} \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} P_k \frac{t^k}{k!},$$

Multiplying (17) by t^n and summing over $n \geqslant 0$ we find

$$\left[L, \sum P_n(t^n/n!)\right] = \left(\sum \lambda_n(t^n/n!)\right) L + \sum P_{n+1}(t^n/n!)$$

which on $\mathscr{V}$ (see (8)) can be written

$$[L, Z] = \partial Z/\partial t \qquad (\text{on } \mathscr{V}).$$

This completes the proof.

Remark. You can see that not only P_0, but *every* P_n $(n \geqslant 0)$ is in $\mathscr{P}$.

EXAMPLE. Heat equation, $L = \partial^2/\partial x^2 \equiv D^2$. If h and k are functions of x then

$$[L, hD + k] = 2h'L + (h'' + 2k')\, D + k''.$$

Thus $2h'$ must be a constant. At the next iteration we see that $2(2k')'$ must be constant. Thereafter there are no restrictions. Thus the perturbations are of the form $(ax + b)\, D + (cx^2 + dx + e)\, id$, i.e., $\mathscr{P}$ is five-dimensional and generated by $\{1, x, x^2, xD, D\}$. The multiplication table of $\mathscr{M}$ is

TABLE I

	1	x	x^2	D	xD	D^2
1	0	0	0	0	0	0
x	0	0	0	-1	$-x$	$-2D$
x^2	0	0	0	$-2x$	$-2x^2$	$-2-4xD$
D	0	1	$2x$	0	D	0
xD	0	x	$2x^2$	$-D$	0	$-2D^2$
D^2	0	$2D$	$2+4xD$	0	$2D^2$	0

It follows that one can solve

$$\partial w/\partial t = (\partial^2 w/\partial x^2) + (ax+b)(\partial w/\partial x) + (cx^2+dx+e)\,w$$

by a process of quadrature from solutions of the heat equation. This process will be carried out in the next paragraph. The algebra $\mathscr{P}$ seems to be related to work of Hida in [4, 5]. See also Miller [16].

Remark. It is natural to ask whether, by repeated applications of this procedure, one can enlarge the class of perturbations. The answer is in the negative, i.e., if $P \in \mathscr{P}$, then perturbation algebra corresponding to $L + P$ is still only $\mathscr{P}$. The proof is a simple application of Proposition 3.

5. Fourier–Mehler Transform

To illustrate the method by which perturbed equations are solved by quadrature, we now work out two examples arising from the heat equation $u_t = D^2 u$. The first example will be the systematic explanation of the trick given in the Introduction. The second is the derivation of the Fourier–Mehler transform, as defined in Hida [4, 5].

It is necessary to work out the group of the heat equation ($L = D^2$). As this is done in references cited in the Introduction we will be brief. The method of (7) yields the following equations

$$\begin{aligned} \alpha_t - 2\beta_x &= 0 \\ \beta_t - \beta_{xx} - 2\gamma_x &= 0 \\ \gamma_t - \gamma_{xx} &= 0 \end{aligned}$$

having solutions

$$\begin{aligned} \alpha &= a + 2bt + 4ct^2 \\ \beta &= d + 2ft + bx + 4ctx \\ \gamma &= g + fx + 2ct + cx^2; \end{aligned}$$

correspondingly,

$$M = a(\partial^2/\partial x^2) + (d + bx)(\partial/\partial x) + (g + fx + cx^2)\, id.$$

Table II gives a description of G through one-parameter subgroups.

TABLE II

Z	$v(s)$ = orbit of a solution u	invariant solution
$\partial/\partial t$	$u(t+s, x)$	linear function of x
$2t(\partial/\partial t + x(\partial/\partial x)$	$u(e^{2s}t, e^s x)$	Gaussian distribution function $= \int g$
$\partial/\partial x$	$u(t, x+s)$	constant
1	$e^s u(t, x)$	zero
$2t(\partial/\partial x) + x$	$u(t, x + 2st)e^{s^2t+sx}$	Gaussian kernel g
$4t^2(\partial/\partial t)+4tx(\partial/\partial x)+x^2$	$u((t/(1-4st)), (x/(1-4st))) \times (e^{sx^2/(1-4st)}(1-4st)^{-1/2})$	$c_1 g + c_2 g'$

It seems worthwhile to explain how invariant solutions can be computed easily. Take, for example, the next to last subgroup in Table II. If a solution is invariant under this subgroup, its initial value is independent of s, so (reading off the second column) $\phi(x)\, e^{sx} = \phi(x)$, which implies $\phi(x)$ = constant times Dirac delta, hence u = constant times Gaussian kernel. The other invariant solutions could be found in a similar manner.

Let us now carry out the first example. We want to solve (3) so

$$M = 2acD^2 - cxD$$

$$Z = (2ac - 2ct)(\partial/\partial t) - cx(\partial/\partial x) \qquad \text{and} \qquad v(s) = e^{sZ}u$$

has to be computed by solving

$$\partial v/\partial s = (2ac - 2ct)(\partial v/\partial t) - cx(\partial v/\partial x)$$
$$v(0) = u$$

where $u_t = D^2 u$, $u(0, x) = \phi(x)$. The solution, obtained by the method of characteristics, is

$$v(s) = u(\tilde{t}(s), \tilde{x}(s))$$

where $\tilde{t}$ and $\tilde{x}$ are the solutions of

$$d\tilde{t}/ds = 2ac - 2c\tilde{t}$$
$$d\tilde{x}/ds = -c\tilde{x}$$

satisfying the initial condition $\tilde{t}(0) = t$, $\tilde{x}(0) = x$. The solutions are

$$\tilde{t}(s) = te^{-2cs} + a(1 - e^{-2cs})$$
$$\tilde{x}(s) = e^{-cs}x.$$

Finally, the solution to (3) (with $s \leftrightarrow t$) is

$$w(s) = v(s)\,|_{t=0} = u(a(1 - e^{-2cs}), e^{-cs}x)$$

as we found previously.

Until now all parameters have been real. For the last example, which the author found in Hida's papers [4, 5], we try to solve

$$\frac{\partial w}{\partial t} = -\frac{i}{2}\frac{\partial^2 w}{\partial x^2} - \frac{i}{2}(1 - x^2)\,w \equiv Mw$$
$$w(0, x) = \phi(x).$$

We have a perturbation of the Schrödinger equation rather than the heat equation. The setup is as in the previous example:

$$Z = \left(-\frac{i}{2} + 2it^2\right)\frac{\partial}{\partial t} + 2ixt\frac{\partial}{\partial x} + \left(-\frac{i}{2} + it + i\frac{x^2}{2}\right)id$$
$$\frac{d\tilde{t}}{ds} = -\frac{i}{2} + 2i\tilde{t}^2, \qquad \frac{d\tilde{x}}{ds} = 2i\tilde{x}\tilde{t}$$

with $\tilde{x}(0) = x$, $\tilde{t}(0) = t$. It is helpful to notice that $\tilde{x}^2/(4\tilde{t}^2 - 1)$ is an invariant of this system. The solutions are

$$\tilde{t} = \frac{t - (i/2)\tan s}{1 - 2it\tan s}$$
$$\tilde{x} = \frac{x}{\cos s - 2it\sin s}.$$

The solution to

$$\partial v/\partial s = Zv$$
$$v(0) = u$$

is given by the method of characteristics as

$$v(s) = e^{-A(s)}u(\tilde{t}(s), \tilde{x}(s))$$

where (at $t = 0$)

$$A(s) = (i/2)\int_0^s \{1 - 2\tilde{t}(r) - \tilde{x}^2(r)\}\,dr$$
$$= (is/2) + \tfrac{1}{2}\log\cos s - (ix^2/2)\tan s.$$

This gives

$$
\begin{aligned}
w(s) &= v(s)\,|_{t=0} \\
&= e^{-i(s/2)}(\cos s)^{1/2}\, e^{ix^2 \tan s/2} u(-(i/2)\tan s, (x/\cos s)) \\
&= (\pi(1 - e^{2is}))^{-1/2} \int_{\mathbb{R}} e^{(x^2+y^2)/(2i\tan s) - xy/(i\sin s)} \phi(y)\, dy.
\end{aligned}
$$

The last formula is obtained by substitution in the usual Gaussian convolution formula for the solution to the heat equation. It makes sense for $\phi \in L^2(\mathbb{R})$, as explained by Kato [7, p. 494]. Note that at time $s = \pi/2$, w becomes the Fourier transform of ϕ. The full w is the Fourier–Mehler transform of ϕ.

6. A Short Table of Algebras

We conclude with a short table of perturbation algebras.

These algebras were computed by the method of Proposition 3. The third example ($L = xD^2$) can be used to simplify some calculations in [3, 6]. Note that we have applied the method to some degenerate operators; the assumption that L is nondegenerate evidently can be relaxed.

TABLE III

L	Generators of $\mathscr{P}$
D^2	$xD, D, 1, x, x^2$
$D_1{}^2 + D_2{}^2$	$x_1D_1 + x_2D_2,\ x_1D_2 - x_2D_1,\ D_1,\ D_2,\ x_1,\ x_2,\ 1,\ x_1{}^2 + x_2{}^2$
xD^2	$x, xD, 1$
x^2D^2	$x\log xD$, xD, $\log x$, $(\log x)^2$, 1
x^3D^2	$xD, x^{-1}, 1$
e^xD^2	$D, e^{-x}, 1$
$D^2 + x^3$	1

Note added in proof. After this paper was written we became aware of work of S. Steinberg ("Applications of the Lie-algebraic formulas of Baker, Campbell, Hausdorff and Zassenhaus to the calculation of explicit solutions of partial differential equations," Univ. of New Mexico, Technical Report No. 314, October 1975) which deals with similar problems but with an emphasis on rearranging exponentials as in Section 1.

Acknowledgment

The author would like to thank K. H. Hofmann for many helpful discussions. Some of the research was supported by a grant from the Tulane University Council on Research, summer of 1974.

REFERENCES

1. G. W. BLUMAN AND J. D. COLE, The general similarity solution of the heat equation, *J. Math. Mech.* **18** (1969), 1025–1042.
2. G. W. BLUMAN AND J. D. COLE, "Similarity Methods for Differential Equations," Springer–Verlag, New York, 1974.
3. W. FELLER, Two singular diffusion problems, *Ann. Math.* **54** (1951), 173–182.
4. T. HIDA, A role of Fourier transform in the theory of infinite dimensional unitary group, *J. Math. Kyoto Univ.* **13** (1972), 203–212.
5. T. HIDA, A probabilistic approach to infinite dimensional unitary group, to appear.
6. S. KARLIN AND J. MCGREGOR, Classical diffusion processes and total positivity, *J. Math. Anal. Appl.* **1** (1960), 163–183.
7. T. KATO, "Perturbation Theory for Linear Operators," Springer–Verlag, New York, 1966.
8. K. MATSCHAT AND E. A. MÜLLER, Über das Auffinden von Ähnlichkeitslosungen partieller Differentialgleichungssysteme unter Benutzung von Transformationsgruppen, mit Anwendung auf Probleme der Strömungphysik, Miszellaneen der Angewandten Mechanik, Berlin, 1962.
9. J. Y. LEFEBVRE AND P. METZGER, Quelques exemples de groupes d'invariance d'équations aux dérivées partielles, *C. R. Acad. Sci. Paris Ser. A* **279** (1974), 165–168.
10. W. MILLER, JR., "Lie Theory and Special Functions," Academic Press, New York, 1968.
11. W. MILLER, JR., Symmetries of differential equations. The hypergeometric and Euler-Darboux equations, *SIAM J. Math. Anal.* **4** (1973), 314–318.
12. W. MILLER, JR., Lie theory and separation of variables. I. Parabolic cylinder coordinates, *SIAM J. Math. Anal.* **5** (1974), 626–643.
13. W. MILLER, JR., Lie theory and separation of variables. II. Parabolic coordinates, *SIAM J. Math. Anal.* **5** (1974), 822–836.
14. W. MILLER, JR. AND E. G. KALNINS, Lie theory and separation of variables. III. The equation $f_{tt} - f_{ss} = \gamma^2 f$, *J. Math. Phys.* **15** (1974), 1025–1032.
15. W. MILLER, JR. AND E. G. KALNINS, Lie theory and separation of variables. IV. The groups $SO(2, 1)$ and $SO(3)$, *J. Math. Phys.* **15** (1974), 1263–1274.
16. W. MILLER, JR. AND E. G. KALNINS, Lie theory and separation of variables. V. The equations $iU_t + U_{xx} = 0$ and $iU_t + U_{xx} - c/x^2U = 0$, *J. Math. Phys.* **10** (1974), 1728–1737.
17. W. MILLER, JR., C. P. BOYER, AND E. G. KALNINS, Lie theory and separation of variables. VI. The equation $iU_t + \Delta_2 U = 0$, to appear.
18. M. J. MORAN AND R. A. GAGGIOLI, Reduction of the number of variables in systems of partial differential equations, with auxiliary conditions, *SIAM J. Appl. Math.* **16** (1968), 202–215.
19. A. J. A. MORGAN, The reduction by one of the number of independent variables in some systems of partial differential equations, *Q. J. Math. Oxford Ser.* **3** (1952), 250–259.
20. L. V. OVSJANNIKOV, "Group Properties of Differential Equations," Novosibirsk, 1962. (Unedited English transl. by G. W. Bluman, 1967.)
21. S. I. ROSENCRANS, Linearity of the group of a parabolic equation, *J. Math. Anal. Appl.*, to appear.
22. L. WEISNER, Generating functions for Hermite functions, *Canad. J. Math.* **11** (1959), 141–147.